Quantum Information Effects

POPL 2022

Chris Heunen Robin Kaarsgaard

January 20, 2022

School of Informatics, University of Edinburgh

chris.heunen@ed.ac.uk
robin.kaarsgaard@ed.ac.uk

Time and space.

Usually not considered: *information cost*.

Information cost

$$inc :: \text{int32} \to \text{int32}$$
$$inc\; n = n + 1$$

$$reset :: \text{int32} \to \text{int32}$$
$$reset\; n = 0$$

What is the information cost of inc and $reset$?

- Applying inc is free (input always recoverable from output).
- Applying $reset$ costs 32 bits of information (input *never* recoverable from output).

But we can't see this in the type signatures.

Idea (James & Sabry, POPL '12):

Classical computation = Classical *reversible* computation + information effects.

$$erase :: b \rightsquigarrow 1 \qquad\qquad create :: 1 \rightsquigarrow b$$

Is there an analogous result for the quantum case?

Classical states: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Probabilistic mix: e.g., $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$

Superposed quantum states: e.g., $\begin{pmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$

Measurement:

- Wave function collapse?
- The *observer effect*: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

Quantum information cost

$$\begin{aligned} \mathit{not} &:: \text{qubit} \to \text{qubit} \\ \mathit{not}\ q &= \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) q \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \end{aligned} \qquad \begin{aligned} \mathit{measure} &:: \text{qubit} \to \text{qubit} \\ \mathit{measure} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &= \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \end{aligned}$$

What is the information cost of not and $\mathit{measure}$?

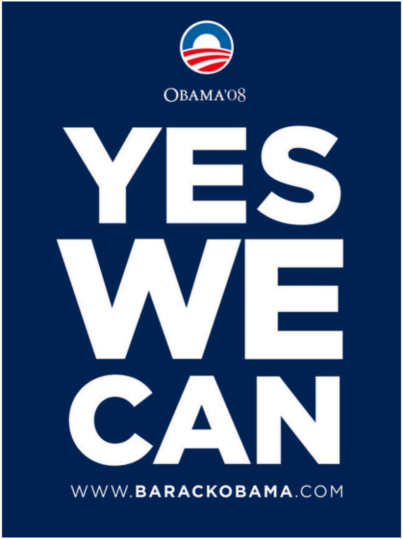
- Applying not is free.
- Applying $\mathit{measure}$ costs 1 qubit of quantum information.

The observer effect destroys quantum information!

James & Sabry, POPL '12: Classical computation is reversible computation with information allocation and erasure.

Can we make sense of *quantum measurement* and *the observer effect* by similar means?

- Is the observer effect a computational effect?
- Are pure states computationally pure?



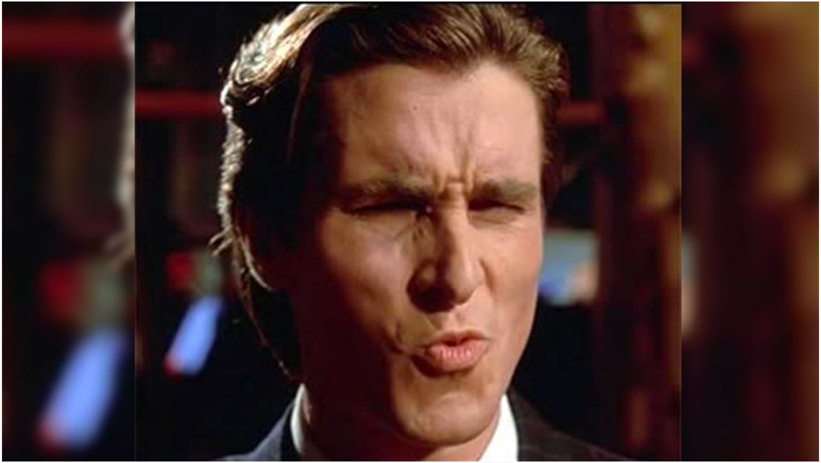
...and yes, they are!

(i) Introduce $\mathcal{U}\Pi$ ("yuppie"), a reversible quantum combinator language (based on the classical combinator language Π).

(ii) Extend $\mathcal{U}\Pi$ with *allocation*, yielding $\mathcal{U}\Pi_a$ ("yuppie-a").

(iii) Extend $\mathcal{U}\Pi_a$ with a *hiding*, yielding $\mathcal{U}\Pi_a^\chi$ ("yuppie-chi-a").

(iv) Argue that $\mathcal{U}\Pi_a^\chi$ can account for measurement.



$\mathcal{U}\Pi$: Unitary Π

Syntax

$$
\begin{aligned}
b &::= 0 \mid 1 \mid b + b \mid b \times b && \text{(base types)} \\
t &::= b \leftrightarrow b && \text{(combinator types)} \\
a &::= id \mid swap^{+} \mid unit^{+} \mid uniti^{+} \mid assoc^{+} \mid associ^{+} \\
&\quad \mid swap^{\times} \mid unit^{\times} \mid uniti^{\times} \mid assoc^{\times} \mid associ^{\times} \\
&\quad \mid distrib \mid distribi \mid distribo \mid distriboi && \text{(primitive combinators)} \\
c &::= a \mid c \fatsemi c \mid c + c \mid c \times c && \text{(combinators)}
\end{aligned}
$$

Typing rules

$$
\begin{array}{rcccl}
id & : & b \leftrightarrow b & : & id \\
swap^{+} & : & b_1 + b_2 \leftrightarrow b_2 + b_1 & : & swap^{+} \\
unit^{+} & : & b + 0 \leftrightarrow b & : & uniti^{+} \\
assoc^{+} & : & (b_1 + b_2) + b_3 \leftrightarrow b_1 + (b_2 + b_3) & : & associ^{+} \\
swap^{\times} & : & b_1 \times b_2 \leftrightarrow b_2 \times b_1 & : & swap^{\times} \\
unit^{\times} & : & b \times 1 \leftrightarrow b & : & uniti^{\times} \\
assoc^{\times} & : & (b_1 \times b_2) \times b_3 \leftrightarrow b_1 \times (b_2 \times b_3) & : & associ^{\times} \\
distrib & : & b_1 \times (b_2 + b_3) \leftrightarrow (b_1 \times b_2) + (b_1 \times b_3) & : & distribi \\
distribo & : & b \times 0 \leftrightarrow 0 & : & distriboi
\end{array}
$$

$$
\frac{c_1 : b_1 \leftrightarrow b_2 \quad c_2 : b_2 \leftrightarrow b_3}{c_1 \fatsemi c_2 : b_1 \leftrightarrow b_3}
\qquad
\frac{c_1 : b_1 \leftrightarrow b_3 \quad c_2 : b_2 \leftrightarrow b_4}{c_1 + c_2 : b_1 + b_2 \leftrightarrow b_3 + b_4}
\qquad
\frac{c_1 : b_1 \leftrightarrow b_3 \quad c_2 : b_2 \leftrightarrow b_4}{c_1 \times c_2 : b_1 \times b_2 \leftrightarrow b_3 \times b_4}
$$

Syntax

$$a ::= \cdots \mid phase_{\varphi} \mid hadamard \qquad \text{(primitive combinators)}$$

Typing rules

$$\begin{array}{rcccl} phase_{\varphi} & : & 1 \leftrightarrow 1 & : & phase_{\overline{\varphi}} \\ hadamard & : & 1+1 \leftrightarrow 1+1 & : & hadamard \end{array}$$

$\mathcal{U}\Pi$ is Π extended with two distinctly quantum combinators, $phase_{\varphi}$ and $hadamard$.

All of these are *unitaries*: reversible quantum computations.

Theorem (Expressivity): $\mathcal{U}\Pi$ is approximately universal for $2^n \times 2^n$ unitaries.

Syntax

$$b ::= 0 \mid 1 \mid b + b \mid b \times b \qquad \text{(base types)}$$

$$t ::= b \rightarrowtail b \qquad \text{(combinator types)}$$

$$c ::= \mathit{lift}\ u \qquad \text{(primitive combinators)}$$

Typing rules

$$\frac{u : b_1 + b_3 \leftrightarrow b_2}{\mathit{lift}\ u : b_1 \rightarrowtail b_2}$$

Extends $\mathcal{U}\Pi$ with the ability to allocate from a hidden heap.

Allocation $alloc : 0 \rightarrowtail b$ implemented as lifted left unitor:

$$\frac{(swap^+ \fatsemi unit^+) : 0 + b \leftrightarrow b}{\mathit{lift}(swap^+; unit^+) : 0 \rightarrowtail b}$$

Extends to an *arrow with choice* over $\mathcal{U}\Pi$, allowing definition of arr, $\ggg$, first, left, etc.

$\mathcal{U}\Pi_a$: $\mathcal{U}\Pi$ WITH ALLOCATION

Using $alloc$ and the other arrow combinators, we can further define a *classical cloning* combinator $clone : b \rightarrowtail b \times b$.

All of these are *isometries*: (roughly) injective quantum computations.

Theorem (Expressivity): $\mathcal{U}\Pi_a$ is approximately universal for $2^n \times 2^m$ isometries.

$\mathcal{U}\Pi_a^\chi$: $\mathcal{U}\Pi$ with hiding and allocation

Syntax

$$b ::= 0 \mid 1 \mid b + b \mid b \times b \qquad \text{(base types)}$$

$$t ::= b \rightsquigarrow b \qquad \text{(combinator types)}$$

$$c ::= \mathit{lift}\ v \qquad \text{(primitive combinators)}$$

Typing rules

$$\frac{v : b_1 \rightarrowtail b_2 \times b_3 \quad b_3 \text{ inhabited}}{\mathit{lift}\ v : b_1 \rightsquigarrow b_2}$$

Extends $\mathcal{U}\Pi_a$ with the ability to discard information to a hidden garbage dump.

Implements $discard : b \rightsquigarrow 1$ as inverse left unitor:

$$\frac{(uniti^{\times} \mathbin{⨾} swap^{\times}) : b \leftrightarrow 1 \times b}{lift(uniti^{\times} \mathbin{⨾} swap^{\times}) : b \rightsquigarrow 1}$$

As with $\mathcal{U}\Pi_a$, this extends to an arrow with choice.

Using $discard$ and the other combinators, we can derive (among other things)

- projections $fst : b \times b' \rightsquigarrow b$ and $snd : b \times b' \rightsquigarrow b'$, and
- measurement $measure : b \rightsquigarrow b$.

All of these are *quantum channels*: arbitrary quantum computations on mixed states (CPTP maps).

Theorem (Expressivity): $\mathcal{U}\Pi_a^\chi$ is approximately universal for quantum channels.

$$measure = clone \ggg fst$$

This aligns with the explanation of measurement offered by *decoherence:*

- $clone$ prepares a new qubit, then applies (reversible) operations to perfectly entangle our qubit with the newly prepared one; then
- when fst is applied, we *forget* one half of the prepared system, from which point on it can be considered no different from any other part of the environment.

This is a precisely a process for leaking information into the environment: no "actual" wave function collapse happens during this process, but having forgotten a part of the system, it appears so from the inside.

Concluding remarks

- Quantum information effects give a type-level separation between quantum programs with and without quantum measurement, and gives an account of measurement through allocation and hiding.
  - Slogan: *The observer effect is a computational effect.*
  - Corollary: *Pure states are computationally pure.*
- Things I didn't mention:
  - Categorical semantics of $\mathcal{U}\Pi$, $\mathcal{U}\Pi_a$, and $\mathcal{U}\Pi_a^\chi$ based on universal constructions applied to rig-categories.
  - Purely categorical statement of Toffoli's *fundamental theorem of reversible computation*.
  - Reasoning about measurement using rig-categories instead of Hilbert spaces.
  - Interpretation of quantum gate sets as well as quantum flow charts (without iteration).
- Read the paper! (artifact also available)
- Happy to chat and answer questions, email me at `robin.kaarsgaard@ed.ac.uk` to book Zoom meeting.