Universal Properties of Partial Quantum Maps

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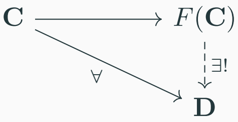
Universal properties in quantum theory



Universal properties in quantum theory

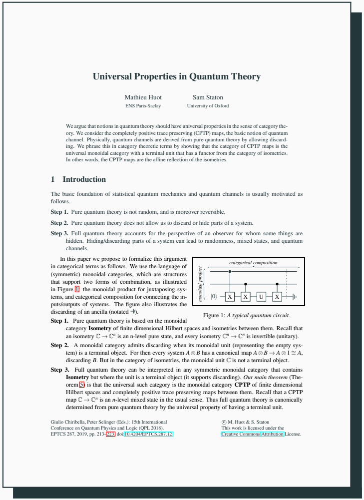
Isolates the *precise features* setting various quantum theories apart.

Free structure provides *syntactic extensions* to programming languages, connecting nicely to the theory of *computational effects*.

Modular semantics for quantum computation.

$$\begin{array}{ccc} \mathbf{S} & \longrightarrow & F(\mathbf{S}) \\ {\scriptstyle [\![-]\!]}\downarrow & & \downarrow{\scriptstyle [\![-]\!]} \\ \mathbf{C} & \longrightarrow & F(\mathbf{C}) \end{array}$$

Universal properties in quantum theory



Universal Properties in Quantum Theory

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We argue that notions in quantum theory should have universal properties in the sense of category theory. We consider the completely positive trace preserving (CPTP) maps, the basic notion of quantum channel. Physically, quantum channels are derived from pure quantum theory by allowing discarding. We phrase this in category theoretic terms by showing that the category of CPTP maps is the universal monoidal category with a terminal unit that has a functor from the category of isometries. In other words, the CPTP maps are the affine reflection of the isometries.

1 Introduction

The basic foundation of statistical quantum mechanics and quantum channels is usually motivated as follows.

Step 1. Pure quantum theory is not random, and is moreover reversible.

Step 2. Pure quantum theory does not allow us to discard or hide parts of a system.

Step 3. Full quantum theory accounts for the perspective of an observer for whom some things are hidden. Hiding/discarding parts of a system can lead to randomness, mixed states, and quantum channels.

In this paper we propose to formalize this argument in categorical terms as follows. We use the language of (symmetric) monoidal categories, which are structures that support two forms of combination, as illustrated in Figure 1: the monoidal product for juxtaposing systems, and categorical composition for connecting the inputs/outputs of systems. The figure also illustrates the discarding of an ancilla (notated ⏚).

categorical composition
monoidal product
$|0\rangle$ X X U X

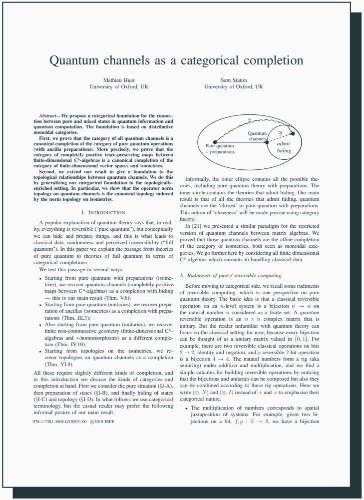
Figure 1: *A typical quantum circuit.*

Step 1. Pure quantum theory is based on the monoidal category **Isometry** of finite dimensional Hilbert spaces and isometries between them. Recall that an isometry $\mathbb{C} \to \mathbb{C}^n$ is an n-level pure state, and every isometry $\mathbb{C}^n \to \mathbb{C}^n$ is invertible (unitary).

Step 2. A monoidal category admits discarding when its monoidal unit (representing the empty system) is a terminal object. For then every system $A \otimes B$ has a canonical map $A \otimes B \to A \otimes 1 \cong A$, discarding B. But in the category of isometries, the monoidal unit $\mathbb{C}$ is not a terminal object.

Step 3. Full quantum theory can be interpreted in any symmetric monoidal category that contains **Isometry** but where the unit is a terminal object (it supports discarding). *Our main theorem* (Theorem 5) is that the universal such category is the monoidal category **CPTP** of finite dimensional Hilbert spaces and completely positive trace preserving maps between them. Recall that a CPTP map $\mathbb{C} \to \mathbb{C}^n$ is an n-level mixed state in the usual sense. Thus full quantum theory is canonically determined from pure quantum theory by the universal property of having a terminal unit.

Giulio Chiribella, Peter Selinger (Eds.): 15th International Conference on Quantum Physics and Logic (QPL 2018).
EPTCS 287, 2019, pp. 213–223, doi:10.4204/EPTCS.287.12



Quantum channels as a categorical completion

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Abstract—**We propose a categorical foundation for the connection between pure and mixed states in quantum information and quantum computation. The foundation is based on distributive monoidal categories.**

First, we prove that the category of all quantum channels is a canonical completion of the category of pure quantum operations (with ancilla preparations). More precisely, we prove that the category of completely positive trace-preserving maps between finite-dimensional C*-algebras is a canonical completion of the category of finite-dimensional vector spaces and isometries.

Second, we extend our result to give a foundation to the topological relationships between quantum channels. We do this by generalizing our categorical foundation to the topologically-enriched setting. In particular, we show that the operator norm topology on quantum channels is the canonical topology induced by the norm topology on isometries.

I. Introduction

A popular explanation of quantum theory says that, in reality, everything is reversible (“pure quantum”), but conceptually we can hide and prepare things, and this is what leads to classical data, randomness and perceived irreversibility (“full quantum”). In this paper we explain the passage from theories of pure quantum to theories of full quantum in terms of categorical completions.

We test this passage in several ways:

- Starting from pure quantum with preparations (isometries), we recover quantum channels (completely positive maps between C*-algebras) as a completion with hiding — this is our main result (Thm. V.6);
- Starting from pure quantum (unitaries), we recover preparation of ancillas (isometries) as a completion with preparations (Thm. III.3);
- Also starting from pure quantum (unitaries), we recover finite non-commutative geometry (finite-dimensional C*-algebras and $*$-homomorphisms) as a different completion (Thm. IV.10);
- Starting from topologies on the isometries, we recover topologies on quantum channels as a completion (Thm. VI.8).

All these require slightly different kinds of completion, and in this introduction we discuss the kinds of categories and completion at hand. First we consider the pure situation (§I-A), then preparation of states (§I-B), and finally hiding of states (§I-C) and topology (§I-D). In what follows we use categorical terminology, but the casual reader may prefer the following informal picture of our main result.

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∀
Quantum channels
∃!
Pure quantum + preparations
admit hiding

Informally, the outer ellipse contains all the possible theories, including pure quantum theory with preparations. The inner circle contains the theories that admit hiding. Our main result is that of all the theories that admit hiding, quantum channels are the ‘closest’ to pure quantum with preparations. This notion of ‘closeness’ will be made precise using category theory.

In [21] we presented a similar paradigm for the restricted version of quantum channels between matrix algebras. We proved that those quantum channels are the affine completion of the category of isometries, both seen as monoidal categories. We go further here by considering all finite dimensional C*-algebras which amounts to handling classical data.

A. Rudiments of pure / reversible computing

Before moving to categorical side, we recall some rudiments of reversible computing, which is one perspective on pure quantum theory. The basic idea is that a classical reversible operation on an n-level system is a bijection $n \to n$ on the natural number n considered as a finite set. A *quantum* reversible operation is an $n \times n$ complex matrix that is unitary. But the reader unfamiliar with quantum theory can focus on the classical setting for now, because every bijection can be thought of as a unitary matrix valued in $\{0, 1\}$. For example, there are two reversible classical operations on bits $2 \to 2$, identity and negation, and a reversible 2-bit operation is a bijection $4 \to 4$. The natural numbers form a rig (aka semiring) under addition and multiplication, and we find a simple calculus for building reversible operations by noticing that the bijections and unitaries can be composed but also they can be combined according to these rig operations. Here we write $(\oplus, N)$ and $(\otimes, I)$ instead of $+$ and $\times$ to emphasise their categorical nature.

- The multiplication of numbers corresponds to spatial juxtaposition of systems. For example, given two bijections on a bit, $f, g : 2 \to 2$, we have a bijection

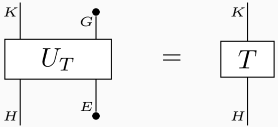
This work:

- Completion of (finite-dimensional Hilbert spaces and) *unitaries* to *contractions* via *Halmos dilation*.
- Completion of finite-dimensional Hilbert spaces and *contractions* to finite-dimensional Hilbert spaces and *CPTN maps* via a variant of *Stinespring dilation*.
- Completion of finite-dimensional Hilbert spaces and *CPTN maps* to finite-dimensional C*-algebras and *CPTN maps* by splitting measurements.

Recall that **Unitary** is a dagger rig groupoid with $(\oplus, O)$ direct sum and $(\otimes, I)$ tensor product.

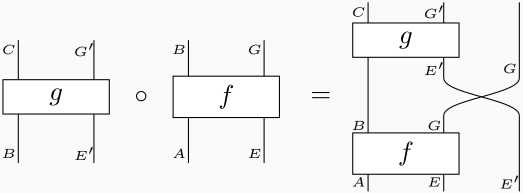
Observation: In **Contraction**, O is a zero object. In **Unitary**, it is neither initial nor terminal.

Theorem (Halmos): Every contraction $T : H \to K$ between finite-dimensional Hilbert spaces extends to a unitary $U_T : H \oplus E \to K \oplus G$ satisfying $T = \pi_K U_T \iota_H$ in an essentially unique way.



Given a dagger rig category **C**, make a new category $LR_\oplus(\mathbf{C})$ with

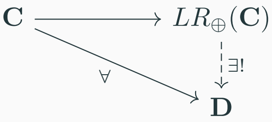
- **Objects:** Those of **C**.
- **Morphisms:** Morphisms $A \to B$ in $LR_\oplus(\mathbf{C})$ are equivalence classes of morphisms $A \oplus E \to B \oplus G$ in **C**.
- **Identities and composition:** Identities are $\mathrm{id}_{A\oplus O}$, composition is



The tensor product $(\otimes, I)$ and direct sum $(\oplus, O)$ even also lift, as does the dagger structure, giving another dagger rig category $LR_\oplus(\mathbf{C})$.

Thanks to the equivalence relation, $LR_{\oplus}(\mathbf{C})$ has the unit of the sum O as a *zero object*.

The evident functor $\mathbf{C} \to LR_{\oplus}(\mathbf{C})$ is even universal among functors into rig categories where the sum unit O is a zero object:



It even works as intended:

Theorem: $LR_{\oplus}(\mathbf{Unitary}) \cong \mathbf{Contraction}$.

Bonus classical result:

Theorem: $LR_{\oplus}(\mathbf{FBij}) \cong \mathbf{FPInj}$.

Let's recall the situation $\mathbf{Isometry} \to \mathbf{FHilb}_{\mathrm{CPTP}}$.

Theorem (Stinespring): Every CPTP map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ in finite dimension extends to an isometry $V : H \to K \otimes E$ (for some finite-dimensional Hilbert space E) such that $\Phi(\rho) = \mathrm{tr}_E(V^\dagger \rho V)$ in an essentially unique way.

Huot and Staton noticed that the trace of a density matrix on H is the unique CPTP map $\mathcal{B}(H) \to \mathbb{C}$, so the tensor unit $\mathbb{C}$ is *terminal* ($\mathbf{FHilb}_{\mathrm{CPTP}}$ is *affine monoidal*).

Theorem (Huot and Staton): $\mathbf{FHilb}_{\mathrm{CPTP}}$ is the *affine completion* of $\mathbf{Isometry}$ as a monoidal category.

Contractions to CPTN maps

In $\mathbf{FHilb}_{\mathrm{CPTN}}$, this approach is *morally correct* but *technically unsound*.

The 0-dimensional Hilbert space is a zero object (i.e., both initial and terminal) in $\mathbf{FHilb}_{\mathrm{CPTN}}$.

The trace is no longer the unique map $\mathcal{B}(H) \to \mathbb{C}$, though it is the unique *trace-preserving map*.

And what about Stinespring?

Contractions to CPTN maps

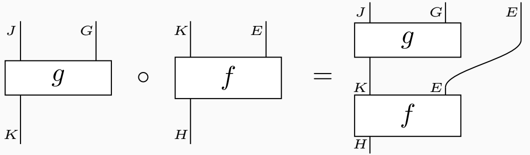
Theorem (Stinespring): Every CPTN map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ in finite dimension extends to a contraction $V : H \to K \otimes E$ (for some finite-dimensional Hilbert space E) such that $\Phi(\rho) = \mathrm{tr}_E(V^\dagger \rho V)$ in an essentially unique way.

Need to be a bit careful with "essentially unique": This is up to an *isometry* applied on the ancilla E, *not a contraction*.

In **Contraction**, the isometries are precisely the *dagger monics*: maps f such that $f^\dagger \circ f = \mathrm{id}$.

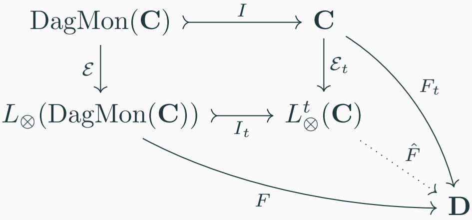
Given a dagger monoidal category $\mathbf{C}$, define a new category $L^t_\otimes(\mathbf{C})$ in the following way:

- **Objects:** Objects of $\mathbf{C}$.
- **Morphisms:** Morphisms $H \to K$ are equivalence classes of morphisms $H \to K \otimes E$.
- **Identities and composition:** Identities are inverse right unitors $\rho_\otimes^{-1}$, composition is



Theorem: $L^t_\otimes(\mathbf{Contraction}) \cong \mathbf{FHilb}_{\mathrm{CPTN}}$.

This construction is universal in that it makes the multiplicative unit I terminal for *total maps* (dagger monics). But it also has a more interesting property…


$\mathrm{DagMon}(\mathbf{C})$
I
$\mathbf{C}$
$\mathcal{E}$
$\mathcal{E}_t$
F_t
$L_{\otimes}(\mathrm{DagMon}(\mathbf{C}))$
I_t
$L^t_{\otimes}(\mathbf{C})$
$\hat{F}$
F
$\mathbf{D}$

$$\begin{array}{ccc} \textbf{Isometry} & \longrightarrow & \textbf{Contraction} \\ \downarrow & & \downarrow \\ \textbf{FHilb}_{\text{CPTP}} & \longrightarrow & \textbf{FHilb}_{\text{CPTN}} \end{array}$$

Hilbert spaces to C*-algebras

Consider a measurement in $\mathbf{FHilb}_{\mathrm{CPTN}}$: an idempotent $\mathcal{B}(H \oplus K) \to \mathcal{B}(H \oplus K)$ of block matrices mapping

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

In $\mathbf{FCstar}_{\mathrm{CPTN}}$, this idempotent splits as a measurement $\mathcal{B}(H \oplus K) \to \mathcal{B}(H) \oplus \mathcal{B}(K)$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (A, D)$$

and a preparation $\mathcal{B}(H) \oplus \mathcal{B}(K) \to \mathcal{B}(H \oplus K)$.

$$(A, D) \mapsto \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

But in $\mathbf{FHilb}_{\mathrm{CPTN}}$, it does not split at all!

Hilbert spaces to C*-algebras

By Artin-Wedderburn, $\mathbf{FCstar}_{\mathrm{CPTN}}$ has direct sums $\bigoplus_{i \in I} \mathcal{B}(H_i)$ of finite-dimensional $\mathcal{B}(H)$s as objects.

$\mathbf{FHilb}_{\mathrm{CPTN}}$ has just finite-dimensional $\mathcal{B}(H)$s as objects.

Observation: For every finite-dimensional C*-algebra $\mathscr{A}$ there exists a finite-dimensional Hilbert space H and a (specifically CPTN) measurement $p : \mathcal{B}(H) \to \mathcal{B}(H)$ such that the image of p is precisely $\mathscr{A}$.

Idea: Encode a finite-dimensional C*-algebra $\mathscr{A}$ as a pair (H, p) of a Hilbert space H and a measurement $p : H \to H$ with $\mathrm{im}(p) = \mathscr{A}$.

- A CPTN map of encoded C*-algebras $(H, p) \to (K, q)$ is a CPTN map $f : \mathcal{B}(H) \to \mathcal{B}(K)$ satisfying $f = q \circ f \circ p$.
- This is the Karoubi envelope (but splitting only measurements)!

For a symmetric monoidal category* $\mathbf{C}$, define a category $\mathbf{Split}_M(\mathbf{C})$ with

- **Objects:** Pairs (H, p) of an object H of $\mathbf{C}$ and a measurement* $p : H \to H$.
- **Morphisms:** Morphisms $(H, p) \to (K, q)$ of $\mathbf{Split}_M(\mathbf{C})$ are morphisms $f : H \to K$ of $\mathbf{C}$ satisfying $q \circ f \circ p = f$.
- **Identities:** The identity $(H, p) \to (H, p)$ is $p : H \to H$.
- **Composition:** As in $\mathbf{C}$.

The evident functor $\mathbf{C} \to \mathbf{Split}_M(\mathbf{C})$ is universal among functors into category where measurements of $\mathbf{C}$ split.

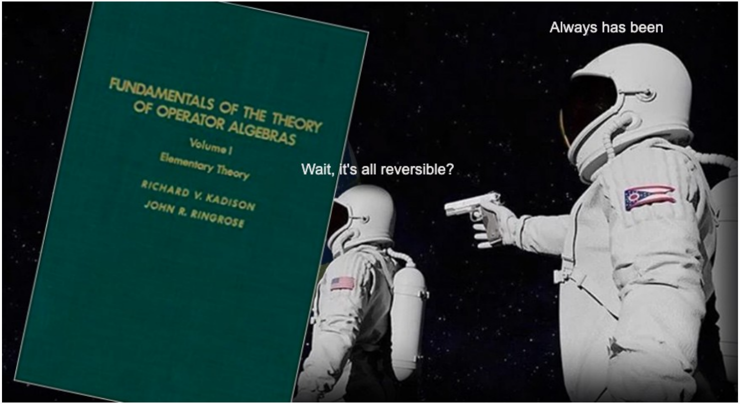
Theorem: $\mathbf{Split}_M(\mathbf{FHilb}_{\mathrm{CPTN}}) \cong \mathbf{FCstar}_{\mathrm{CPTN}}$.

Concluding remarks

Universal properties isolate the precise features setting various quantum theories apart.

Universal constructions provide mechanical extensions to programming languages, along with extensible program semantics.

Macron reacts


FUNDAMENTALS OF THE THEORY
OF OPERATOR ALGEBRAS
Volume I
Elementary Theory
RICHARD V. KADISON
JOHN R. RINGROSE
Wait, it's all reversible?
Always has been

The actual meme abstract